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Optimal analysis of the performance of an irreversible quantum heat engine with spin systems

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Abstract

It is considered that the cycle of a quantum heat engine using many non-interacting spin-$1/2$ systems as the working substance is composed of two adiabatic and two isomagnetic field processes and is referred to as a spin quantum Brayton engine cycle. Based on the quantum master equation and semi-group approach, expressions for the efficiency and power output of the cycle are derived. By using numerical solutions, the power output of the heat engine subject to finite cycle duration is optimized. The maximum power output and the corresponding parameters are calculated numerically. The optimal region of the efficiency and the optimal ranges of temperatures of the working substance and times spent on the two isomagnetic field processes are determined, so that the general optimum performance characteristics of the cycle are revealed. Moreover, the optimal performance of the cycle in the high-temperature limit is also analysed in detail. The results obtained here are further generalized, so that they may be directly used to describe the performance of a quantum Brayton heat engine using spin-J systems as the working substance.

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1. Introduction

The investigation on the performance characteristics of thermodynamic cycles has been widely extended from classical to quantum cycles [1–19]. Quantum cycles have become interesting research subjects. For example, Scully *et al* analysed the performance of a quantum heat engine operating at the radiation pressure from a single mode radiation field which drives a piston engine or a photon Carnot engine and pointed out that the phase associated with the atomic coherence provides a new control parameter, which can be varied to increase the

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temperature of the radiation field and to extract work from a single heat bath, while the real physics behind the second law of thermodynamics is not violated [4, 17, 18]. Some authors have intensively studied the influence of several factors on the performance characteristics of quantum thermodynamic cycles working with spin systems [1–3] or harmonic oscillator systems [9–12], based on the quantum master equation and semi-group approach. The maximum power output or cooling rate of these cycles and the corresponding performance parameters were calculated. In addition, Feldmann and Kosloff [19] investigated the optimal performance of the quantum heat engine and heat pump working with spin-$1/2$ systems by using the dynamical model which is based on the probability distribution of occupancy of the evolving energy levels and a Pauli master equation. The power output of the engine is optimized with respect to time allocation between the contact time with the hot and cold baths as well as the adiabats.

Quantum cycle models of heat engines and refrigerators show a remarkable similarity to thermodynamic cycles obeying macroscopic dynamics. The Carnot efficiency provides an upper bound on the efficiencies of quantum heat engines operating between two heat reservoirs. The irreversible operation of quantum engines with finite power output [1, 3, 9] has many similarities to macroscopic endoreversible engines. Consequently, the investigation related to some new quantum engines will be helpful not only to understand deeply the performance characteristics of the quantum thermodynamic cycles but also to reveal further the relationship and distinguish between the quantum and the corresponding classical thermodynamic cycles.

Similar to classical thermodynamic cycles, quantum thermodynamic cycles may have different typical cycle models. For example, when spin systems are used as the working substance, there may be the quantum Carnot cycle [3, 16] consisting of two isothermal and adiabatic (i.e., constant-polarization) processes, the Ericsson cycle consisting of two isothermal and two isomagnetic field processes [1, 2], and Brayton cycle consisting of two adiabatic (i.e., constant-polarization) and two isomagnetic field processes, etc. The optimal performance of the quantum Carnot cycle and the Ericsson cycle has been investigated and many significant results have been obtained [1–3]. However, the optimal performance of the quantum Brayton cycle working with spin systems has been rarely studied. Therefore, it is of great significance to optimize the performance of this quantum thermodynamic cycle.

In the present paper, a new cycle model of a quantum mechanical heat engine using many non-interacting spin-$1/2$ systems as the working substance and consisting of two adiabatic and two isomagnetic field processes is established. The general expressions of the efficiency and power output of the cycle are derived, based on the dynamical semi-group approach of the quantum theory of open systems. The important performance parameters such as the efficiency, power output and the temperatures of the working substance are optimized and the general performance characteristics of the cycle are analysed. The optimally operating regions of some performance parameters in the heat engine are determined. The results obtained here are different from those derived from the quantum Carnot and Ericsson cycles.

2. A spin quantum Brayton engine cycle

We first consider a quantum heat engine operating between two heat reservoirs at constant temperatures T_h and T_c, in which the spin systems used as the working substance are not only coupled mechanically with the given 'magnetic field' $\omega(t)$, but also with a heat reservoir at temperature T. Based on the quantum mechanics, the Hamiltonian of the interaction between a magnetic moment $\mathbf{M}$ and a magnetic field $\mathbf{B}$ is given by $\hat{\mathbf{H}}(t) = -\hat{\mathbf{M}} \cdot \mathbf{B}$, where the magnitude of the magnetic field can change over time, but is not allowed to reach zero. For a single-spin quantum system, the magnetic moment $\mathbf{M}$ is proportional to the spin angular momentum $\mathbf{S}$.

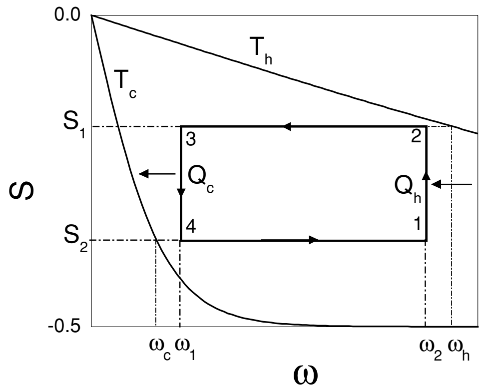

Figure 1. The S–ω diagram of an irreversible quantum heat engine cycle, where the unit of ω is joules.

When the direction of the magnetic field **B** is chosen constant and along the positive z-axis, the Hamiltonian may be given by

$$\hat{\mathbf{H}}(t) = 2\mu_B \hat{\mathbf{S}} \cdot \mathbf{B} = 2\mu_B B_z(t)\hat{\mathbf{S}}_z, \tag{1}$$

where μ_B is the Bohr magneton. Throughout this paper we define $\omega(t) = 2\mu_B B_z(t)$ for simplicity. Obviously, ω is positive since the spin angular momentum and magnetic moment are in opposite directions. As described in [3], one can refer to ω rather than B_z as 'the field'. Therefore, the Hamiltonian of an isolated single spin-1/2 system in the presence of the field $\omega(t)$ may be expressed as

$$\hat{\mathbf{H}}(t) = \omega(t)\hat{\mathbf{S}}_z. \tag{2}$$

Based on the statistical mechanics, the expectation value of a spin-1/2 angular momentum $\hat{S}_z$ is given by

$$S = \langle \hat{\mathbf{S}}_z \rangle = -\tfrac{1}{2}\tanh(\beta'\omega/2), \tag{3}$$

where $\beta' = 1/T'$ (T' is the internal temperature of the quantum system in energy units), $-1/2 < S < 0$ (our units are such that $\hbar = 1$). For the convenience of writing, 'temperature' will refer to β' (or β) rather than T' (or T). Using equation (3), one can plot the S–ω diagram of a quantum heat engine cycle consisting of two adiabatic and two isomagnetic field processes, as shown schematically in figure 1, where ω_h is the upper bound of ω in the high-isomagnetic field processes and ω_c is the lower bound of ω in the low-isomagnetic field processes. In the cycle, the two adiabatic processes $S = S_1$ and $S = S_2$ are connected by the two isomagnetic field processes $\omega = \omega_1$ and $\omega = \omega_2$ with $\omega_2 > \omega_1$. In the two isomagnetic field processes, the spin systems are, respectively, coupled with the heat reservoirs at constant temperatures $\beta = \beta_h$ and $\beta = \beta_c$, and the amounts of heat exchange between the working substance and the heat reservoirs are represented by Q_h and Q_c.

In order to understand better the cycle mechanism of the quantum heat engine, we further describe each of the four processes in the cycle. On the first process $1 \to 2$, the working substance is coupled with the heat reservoir at 'temperature' β_h for period t_h, while the energy gap is kept fixed at the value ω_2. The conditions are such that the internal temperature of the working substance is lower than β_h. Consequently, population transfer is induced from the lower level to the upper one, thereby diminishing the population difference between the two levels and making S less negative, i.e., in this process, the polarization is changing from the initial polarization S_2 to the polarization S_1. Because ω is kept fixed, no work is done. The energy transfer is equal to the amount of heat Q_h absorbed by the working substance. In the second process (adiabatic process) $2 \to 3$, the working substance is decoupled from the heat reservoir for period t_{23}, and the energy gap is varied from ω_2 to ω_1, while adiabaticity dictates that no change in probabilities should occur, so that the population is kept fixed at the value S_1, i.e. the populations of the two levels are not changed during this process. Decreasing ω adiabatically reduces the energy gap between the two energy levels. Thus, no heat transfer is involved and the only form of energy exchange is the work done on the working substance by the external field. The third process, $3 \to 4$ is similar to the first. The working substance is now coupled with a heat reservoir at 'temperature' β_c for period t_c and the energy gap is kept fixed at the value ω_1. Population transfer from the upper to the lower level is induced, hence restoring the population difference between the two levels and making S more negative. The coupling with the heat reservoir is maintained until the original value of $S = S_2$ is restored. The only form of energy exchange is the heat Q_c flowing out of the working substance and into the heat reservoir. The fourth process, $4 \to 1$, closes the cycle, and is similar to the second process. The working substance is decoupled from the low-temperature heat reservoir for period t_{41}, and the energy gap is restored to its original value, ω_2; however the population in the two levels remain fixed at $S = S_2$. This process involves no heat transfer, and the only form of energy exchange is the work done by the working substance on the surrounding.

Due to finite-rate heat transfer between the working substance and the heat reservoirs, the temperatures $\beta_1, \beta_2, \beta_3$ and β_4 of the working substance in states 1, 2, 3 and 4 are different from those of the heat reservoirs and there is a relation $\beta_c \geqslant \beta_4 > \beta_2 \geqslant \beta_h$. From equation (3) and figure 1, one can find

$$\beta_3\omega_1 = \beta_2\omega_2 \tag{4}$$

and

$$\beta_4\omega_1 = \beta_1\omega_2. \tag{5}$$

From equations (4) and (5), we obtain an important relation $\beta_1/\beta_2 = \beta_4/\beta_3$, which restricts the temperatures of the four states 1, 2, 3 and 4 in figure 1. It is of interest to note that the important relation is identical with that of a Brayton heat engine working with an ideal gas and the two isomagnetic field processes in the spin quantum heat engine correspond to the two constant-pressure processes in a gas Brayton refrigeration cycle, so that the quantum engine cycle shown in figure 1 may reasonably be referred to as the spin quantum Brayton engine cycle.

When the spin system mentioned above are used as the working substance of a quantum heat engine, the internal energy of the working substance may change by changing either the 'magnetic field' ω or the polarization of the spin system S. Using equation (2), one can obtain the internal energy of the spin system

$$E = \langle \hat{\mathbf{H}} \rangle = \omega(t)\langle \hat{\mathbf{S}}_z \rangle = \omega S \tag{6}$$

and its differential form

$$\mathrm{d}E/\mathrm{d}t = S\,\mathrm{d}\omega/\mathrm{d}t + \omega\,\mathrm{d}S/\mathrm{d}t. \tag{7}$$

The operation of a quantum cycle is followed through the changes in the observables of the working fluid. Based on the semi-group formalism, the equation of motion of an operator $\hat{\mathbf{X}}$ in the Heisenberg picture is given by the quantum master equation [3, 20–22], i.e.,

$$\frac{\mathrm{d}\hat{\mathbf{X}}}{\mathrm{d}t} = \mathrm{i}[\hat{\mathbf{H}}, \hat{\mathbf{X}}] + \frac{\partial \hat{\mathbf{X}}}{\partial t} + L_D(\hat{\mathbf{X}}), \tag{8}$$

where $L_D(\hat{\mathbf{X}}) = \sum_\alpha \gamma_\alpha \big(\hat{\mathbf{V}}_\alpha^+[\hat{\mathbf{X}}, \hat{\mathbf{V}}_\alpha] + [\hat{\mathbf{V}}_\alpha^+, \hat{\mathbf{X}}]\hat{\mathbf{V}}_\alpha\big)$ is a dissipation term and originates from a thermal coupling of the spin with a heat reservoir, $\mathbf{V}_\alpha^+$ and $\mathbf{V}_\alpha$ are operators in the Hilbert space of the system and are Hermitian conjugates, and γ_α are phenomenological positive coefficients. Replacing $\hat{\mathbf{X}}$ by $\hat{H}$ in equation (8), one can obtain the rate of change of the internal energy as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{H}\rangle = \left\langle \frac{\partial \hat{H}}{\partial t}\right\rangle + \langle L_D(\hat{H})\rangle. \tag{9}$$

Combining equations (7) and (9) and comparing them with the differential form of the first law of thermodynamics, $\mathrm{d}E/\mathrm{d}t = \mathrm{d}W/\mathrm{d}t + \mathrm{d}Q/\mathrm{d}t$, one can easily find that the instantaneous power and heat flow [3, 19] are $P = \mathrm{d}W/\mathrm{d}t = \langle \partial \hat{H}/\partial t\rangle = S\,\mathrm{d}\omega/\mathrm{d}t$ and $\mathrm{d}Q/\mathrm{d}t = \langle L_D(\hat{H})\rangle = \omega\,\mathrm{d}S/\mathrm{d}t$, respectively. Hence, the work and heat inexact differentials are now given by

$$\mathrm{d}W = S\,\mathrm{d}\omega \tag{10}$$

and

$$\mathrm{d}Q = \omega\,\mathrm{d}S. \tag{11}$$

It should be pointed out that although the working substance of the cycle in the heat engine model is the same as those in [1–3], the cycle mode is different so that some new results can be obtained.

3. Efficiency and power output

The efficiency and power output are two important performance parameters which are often considered in the optimal design and theoretical analysis of heat engine. Using equation (11), one can find that the amounts of heat exchange between the working substance and the heat reservoirs in the two isomagnetic field processes mentioned above are, respectively, given by

$$Q_h = Q_{12} = \int_{S_2}^{S_1} \omega_2\,\mathrm{d}S = \omega_2(S_1 - S_2) \tag{12}$$

and

$$Q_c = Q_{34} = \int_{S_1}^{S_2} \omega_1\,\mathrm{d}S = \omega_1(S_2 - S_1). \tag{13}$$

Using equations (4), (5), (12) and (13), one can find that the expressions for the efficiency η and power output P are, respectively, given by

$$\eta = \frac{Q_h + Q_c}{Q_h} = 1 - \frac{\omega_1}{\omega_2} = 1 - \frac{\tanh^{-1}(2S_2)}{\tanh^{-1}(2S_1)}\frac{\beta_2}{\beta_4} \tag{14}$$

and

$$P = \frac{Q_h + Q_c}{t} = \frac{(S_1 - S_2)(\omega_2 - \omega_1)}{t}, \tag{15}$$

where t is the cycle period.

In order to analyse the performance characteristics of the spin quantum Brayton heat engine, the times of the heat-exchange processes have to be calculated. To this end, we begin to solve the equation of motion that determines the time evolution of the spin angular momentum. For a spin system, $\mathbf{V}_\alpha^+$ and $\mathbf{V}_\alpha$ may be chosen to be the spin annihilation and creation operators: $\hat{\mathbf{S}}_- = \hat{\mathbf{S}}_x - \mathrm{i}\hat{\mathbf{S}}_y$ and $\hat{\mathbf{S}}_+ = \hat{\mathbf{S}}_x + \mathrm{i}\hat{\mathbf{S}}_y$, and $\hat{\mathbf{H}} = \omega\hat{\mathbf{S}}_z$. Substituting $\hat{\mathbf{S}}_-$, $\hat{\mathbf{S}}_+$, $\hat{\mathbf{H}}$ and $\hat{\mathbf{X}} = \hat{\mathbf{S}}_z$ into equation (8), one can prove [3] that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -a\,\mathrm{e}^{q\beta\omega}[2(1 + \mathrm{e}^{\beta\omega})S + \mathrm{e}^{\beta\omega} - 1], \tag{16}$$

where $a > 0$, $-1 < q < 0$ and β, ω and S are, in general, dependent on time [3]. The solution of equation (16) is given by

$$t = -\frac{1}{a}\int_{S_i}^{S_f} \frac{\mathrm{d}S}{\mathrm{e}^{q\beta\omega}[2(1+\mathrm{e}^{\beta\omega})S + \mathrm{e}^{\beta\omega} - 1]}, \tag{17}$$

where S_i and S_f are the initial and finial values of S along a given path $S(\beta', \omega)$. Equation (17) is a general expression of time evolution for a spin-1/2 system coupling with the heat reservoir and the external magnetic field.

Using equations (3) and (17), one can calculate the times spent on the four processes in the cycle. Substituting $S_i = S_2$, $S_f = S_1$, $\beta = \beta_h$ and $\omega = \omega_2$ into equation (17), one obtains the time of the one isomagnetic field process as

$$t_h = \frac{1}{2a\,\mathrm{e}^{q\beta_h\omega_2}(1+\mathrm{e}^{\beta_h\omega_2})} \ln\left[\frac{2(1+\mathrm{e}^{\beta_h\omega_2})S_2 + \mathrm{e}^{\beta_h\omega_2} - 1}{2(1+\mathrm{e}^{\beta_h\omega_2})S_1 + \mathrm{e}^{\beta_h\omega_2} - 1}\right]. \tag{18}$$

Similarly, substituting $S_i = S_1$, $S_f = S_2$, $\beta = \beta_c$ and $\omega = \omega_1$ into equation (17), one obtains the time of the other isomagnetic field process as

$$t_c = \frac{1}{2a\,\mathrm{e}^{q\beta_c\omega_1}(1+\mathrm{e}^{\beta_c\omega_1})} \ln\left[\frac{2(1+\mathrm{e}^{\beta_c\omega_1})S_1 + \mathrm{e}^{\beta_c\omega_1} - 1}{2(1+\mathrm{e}^{\beta_c\omega_1})S_2 + \mathrm{e}^{\beta_c\omega_1} - 1}\right]. \tag{19}$$

It can be seen from equation (11) that for the two adiabatic processes in the cycle, $\mathrm{d}Q = 0$ and consequently $\mathrm{d}S = 0$. This implies that the times spent on the two adiabatic processes are very small compared with those spent on the isomagnetic field processes. For the sake of calculative convenience, it may be assumed that they are proportional to the times spent on the isomagnetic field processes. Thus, the cycle period is given by

$$t = t_c + t_h + t_{23} + t_{41} = (1+\gamma)(t_c + t_h), \tag{20}$$

where γ is a proportional constant.

Starting from equations (14), (15) and (18)–(20), one can optimize the important performance parameters of a spin quantum Brayton heat engine.

4. Optimum performance characteristics

Using equations (15) and (18)–(20), one can plot the graph of the dimensionless power output varying with 'the fields' (ω_1, ω_2) for given $q, \beta_c, \beta_h, \omega_h$ and ω_c, as shown in figure 2, where $P^* = P(1+\gamma)/(2a\omega_c)$ is the dimensionless power output. It can be seen from figure 2 that there is a maximum for the power output. Using equation (15) and the extremal condition $\partial P/\partial\omega_1 = 0$, we can obtain the following equation

$$\frac{t_h + t_c}{C\beta_c} - (\omega_2 - \omega_1)\left\{\frac{4(S_1 - S_2)\,\mathrm{e}^{\beta_c\omega_1}}{AB} - \frac{(q+1)\,\mathrm{e}^{\beta_c\omega_1} + q}{\mathrm{e}^{\beta_c\omega_1} + 1}\ln\left(\frac{A}{B}\right)\right\} = 0, \tag{21}$$

where

$$C = 1/[2a\,\mathrm{e}^{q\beta_c\omega_1}(1+\mathrm{e}^{\beta_c\omega_1})], \qquad A = [2(1+\mathrm{e}^{\beta_c\omega_1})S_1 + \mathrm{e}^{\beta_c\omega_1} - 1]$$

and

$$B = [2(1+\mathrm{e}^{\beta_c\omega_1})S_2 + \mathrm{e}^{\beta_c\omega_1} - 1].$$

Equation (21) gives an optimal relation between $\beta_2(\omega_2)$ and $\beta_4(\omega_1)$ for given $q, \beta_c, \beta_h, \omega_h$ and ω_c, but it is too complicated to yield a simple analytical solution. However, for given $q, \beta_h, \beta_c, \omega_h$ and ω_c, the P^*–η optimal characteristic curve can be plotted by using equations (14), (15) and (21), as shown in figure 3. In the figure, the parameters $kT_h = 5.0J$, $kT_c = 1.0J$, $\omega_h = 5.0J$, $\omega_c = 2.0J$ and $q = -0.5$ are adopted [3].

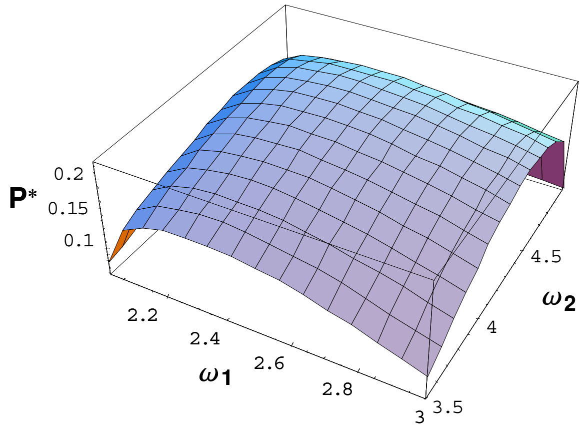

Figure 2. The dimensionless power output $P^* = P(1+\gamma)/(2a\omega_c)$ as a function of the fields (ω_1, ω_2). The graph is drawn for the parameters $kT_h = 5.0J, kT_c = 1.0J, \omega_h = 5.0J, \omega_c = 2.0J, q = -0.5$ and $\hbar = 1$.

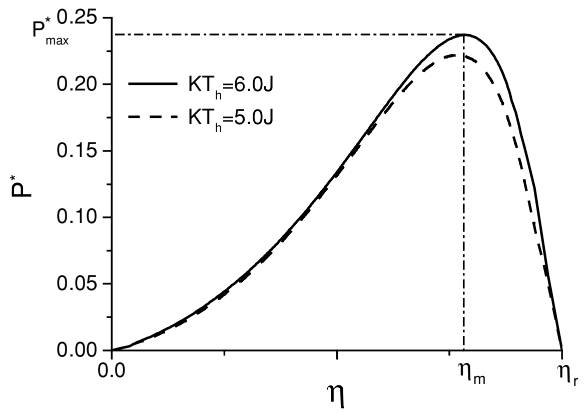

Figure 3. The dimensionless power output $P^* = P(1+\gamma)/(2a\omega_c)$ versus the efficiency η. The solid and dashed curves are drawn for $kT_h = 6.0J$ and $kT_h = 5.0J$, respectively. The values of the parameters $kT_c, \omega_h, \omega_c, q$ and $\hbar$ are the same as those used in figure 2.

Figure 3 shows the dimensionless power output $P(1+\gamma)/(2a\omega_c)$ as a function of the efficiency η for an irreversible spin quantum Brayton heat engine. It is seen from the curves in figure 3 that there exists a maximum power output $P_{\max}$ and a corresponding efficiency η_m for a set of given parameters $q, \beta_h, \beta_c, \omega_h$ and ω_c. Obviously, for different given parameters, the maximum power output $P_{\max}$ and corresponding efficiency η_m will be different. For example, for given 'the field' ratio ω_c/ω_h, the less the temperature ratio T_c/T_h of the two heat reservoirs, the larger the maximum power output and the corresponding efficiency; for given T_c/T_h, the less 'the field' ratio ω_c/ω_h of the external magnetic field, the larger the maximum power output and corresponding efficiency; as indicated in table 1. It is also seen from the curves in figure 3 that when $P < P_{\max}$, there are two different efficiencies for a given power output P, where one is smaller than η_m and the other is larger than η_m. When $\eta < \eta_m$, the power output decreases as the efficiency decreases. It is thus clear that the region of $\eta < \eta_m$ is not optimal for a spin quantum Brayton heat engine. Consequently, the optimal region of the efficiency

Table 1. Optimal parameters at the maximum power output for given T_c/T_h, ω_c/ω_h and $q = -0.5$.

T_c/T_h	ω_c/ω_h	$P^*_{\max}$	η_m	T_{1m}/T_{3m}	ω_{1m}/ω_{2m}
0.2	0.4	0.2219	0.4599	0.2701	0.5401
	0.6	0.1680	0.3232	0.2256	0.6768
0.3	0.4	0.0972	0.4104	0.4913	0.5896
	0.6	0.0909	0.3046	0.3863	0.6954
0.4	0.4	0.0750	0.3648	0.6384	0.6352
	0.6	0.0710	0.2932	0.4736	0.7068

should be

$$\eta_m \leqslant \eta < \eta_r, \tag{22}$$

where $\eta_r = 1 - \omega_c/\omega_h$ is the maximum efficiency of a spin Brayton heat engine. When a spin quantum Brayton heat engine is operated in this region, the power output will increase as the efficiency decreases, and vice versa. This shows that η_m is an important parameter for a spin quantum Brayton heat engine. It determines the allowable value of the lower bound of the optimal efficiency.

Using equation (22) and the above results, we can further find that the optimal ranges of the highest and lowest 'temperatures' β_2 and β_4 of the working substance in the two isomagnetic field processes should be

$$\beta_{2m} \geqslant \beta_2 > \beta_h \tag{23}$$

and

$$\beta_{4m} \leqslant \beta_4 < \beta_c, \tag{24}$$

where the values of β_{2m} and β_{4m} can be calculated from equations (15) and (18)–(21) and have been listed in table 1. From equations (4), (5), (23) and (24), we can obtain the optimal ranges of the 'temperatures' of the working substance at states 1 and 3 as

$$\beta_{1m} = \beta_{4m}\omega_1/\omega_2 \leqslant \beta_1 < \beta_c\omega_1/\omega_2, \tag{25}$$

and

$$\beta_{3m} = \beta_{2m}\omega_2/\omega_1 \geqslant \beta_3 > \beta_h\omega_2/\omega_1. \tag{26}$$

It is obvious that the parameters $\beta_{1m}, \beta_{2m}, \beta_{3m}, \beta_{4m}$ are very important for a spin quantum Brayton heat engine and equations (23)–(26) provide four significant criteria for the selection of optimally operating conditions.

In addition, using equations (18), (19), (21) and (22), one can find that the times spent on the two isomagnetic field processes, t_1 and t_2, of the cycle should be controlled to satisfy the following conditions:

$$t_h \geqslant t_{hm}, \tag{27}$$

and

$$t_c \geqslant t_{cm}, \tag{28}$$

where

$$t_{hm} = \frac{1}{2a\,e^{q\beta_h\omega_{2m}}(1 + e^{\beta_h\omega_{2m}})} \ln\left[\frac{2(1 + e^{\beta_h\omega_{2m}})S_2 + e^{\beta_h\omega_{2m}} - 1}{2(1 + e^{\beta_h\omega_{2m}})S_1 + e^{\beta_h\omega_{2m}} - 1}\right]$$

and

$$t_{cm} = \frac{1}{2a\,e^{q\beta_c\omega_{1m}}(1 + e^{\beta_c\omega_{1m}})} \ln\left[\frac{2(1 + e^{\beta_c\omega_{1m}})S_1 + e^{\beta_c\omega_{1m}} - 1}{2(1 + e^{\beta_c\omega_{1m}})S_2 + e^{\beta_c\omega_{1m}} - 1}\right].$$

If not, the quantum heat engine could not be operated in the rational region.

5. Discussion

(1) When the temperatures of two heat reservoirs are high enough, i.e., $\beta\omega \ll 1$, the results obtained above can be simplified. For example, equations (3), (15), (18), (19) and (21) can be, respectively, simplified as

$$S = -\frac{1}{4}\beta'\omega, \tag{29}$$

$$P = \frac{a(\omega_2 - \omega_1)(\beta_4\omega_1 - \beta_2\omega_2)}{(1+\gamma)\ln\{[(\beta_h\omega_2 - \beta_4\omega_1)(\beta_c\omega_1 - \beta_2\omega_2)]/[\omega_1\omega_2(\beta_h - \beta_2)(\beta_c - \beta_4)]\}}, \tag{30}$$

$$t_h = \frac{1}{4a}\ln\left(\frac{\beta_h\omega_2 - \beta_4\omega_1}{\beta_h\omega_2 - \beta_2\omega_2}\right), \tag{31}$$

$$t_c = \frac{1}{4a}\ln\left(\frac{\beta_c\omega_1 - \beta_2\omega_2}{\beta_c\omega_1 - \beta_4\omega_1}\right), \tag{32}$$

and

$$\ln\left[\frac{(\beta_h\omega_2 - \beta_4\omega_1)(\beta_c\omega_1 - \beta_2\omega_2)}{\omega_1\omega_2(\beta_h - \beta_2)(\beta_c - \beta_4)}\right] - \frac{\beta_c(\beta_4\omega_1 - \beta_2\omega_2)(\omega_2 - \omega_1)}{\omega_1(\beta_c\omega_1 - \beta_2\omega_2)(\beta_c - \beta_4)} = 0. \tag{33}$$

(2) When the working substance is composed of spin-J system ($J = 1/2, 1, 3/2, 2, \ldots$), the mean value of the spin angular momentum can be written as [23, 24]

$$S = \langle \hat{\mathbf{S}}_z \rangle = -J B_J(\beta'\omega J), \tag{34}$$

where $-J \leqslant S \leqslant J$ and $B_J(x) = [(2J+1)/(2J)]\coth[(2J+1)x/(2J)] - (1/2J)\coth[x/(2J)]$ is the Brillouin function. Based on the quantum master equation and semi-group approach, one can prove that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -2a\,\mathrm{e}^{q\beta\omega}\{(1+\mathrm{e}^{\beta\omega})S + (\mathrm{e}^{\beta\omega} - 1)[J(J+1) - M]\}, \tag{35}$$

where $M = \langle \hat{S}_z^2 \rangle$ and $\langle \hat{S}^2 \rangle = J(J+1)$. Solving equation (35), we obtain the general expression of time evolution for a spin-J system coupling with the heat reservoir and the external magnetic field as

$$t = -\frac{1}{a}\int_{S_i}^{S_f} \frac{\mathrm{d}S}{\mathrm{e}^{q\beta\omega}\{2(1+\mathrm{e}^{\beta\omega})S + 2[J(J+1) - M](\mathrm{e}^{\beta\omega} - 1)\}}. \tag{36}$$

Using the similar method mentioned above and equations (14), (15) and (36), we can analyse the performance characteristics of the quantum heat engine working with spin-J systems. At high temperatures, equations (34) and (36) may be, respectively, simplified as

$$S = -\frac{J(J+1)}{3}\beta'\omega \tag{37}$$

and

$$t = -\frac{1}{2a}\int_{S_i}^{S_f} \frac{\mathrm{d}S}{2S + \beta\omega[J(J+1) - M]}, \tag{38}$$

where $M = J(J+1)/3$. Comparing equation (37) with equation (29), we can find that the amount of heat of the two isomagnetic field processes in the cycle may be obtained by multiplying the factor of $4J(J+1)/3$ in equations (12) and (13). On the other hand, we can find form equations (37) and (38) that the times of the two isomagnetic field processes are the same as equations (31) and (32). Thus, at high temperatures, the efficiency of the quantum cycle using the spin-J systems as the working substance is the same as that of the quantum cycle working with the spin-1/2 systems, while the power output is $4J(J+1)/3$ times the quantum cycle using the spin-1/2 systems as the working substance.

(3) The above discussion only refers to a single spin-J system. For the working substance consisting of many non-interacting spin-J systems, the efficiency is still true, while the work output, power output and amounts of heat exchange can be obtained as long as the above results are simply multiplied by the total number of spin systems.

6. Conclusions

In this study, the quantum heat engine using many non-interacting spin systems as the working substance and consisting of two adiabatic and two isomagnetic field processes may reasonably be referred to as the spin quantum Brayton heat engine. It is one of the three important quantum thermodynamic cycle models working with spin systems. Based on the statistical mechanics, semi-group formalism and equation of motion that determines the time evolution of the spin angular momentum, we have derived the general expressions of some important parameters of the quantum heat engine. By using numerical solutions, the performances of the spin quantum Brayton heat engine are optimized. Several optimum performance curves have been presented for a set of given parameters. The optimally operating regions of some important performance parameters are determined. The general performance characteristics of the quantum heat engine are revealed. Finally, the results obtained are further generalized, so that they are also suitable for the working substance consisting of non-interacting spin-J systems.

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